FINITE ELEMENT MODEL UPDATE VIA BAYESIAN ESTIMATION AND MINIMIZATION OF DYNAMIC RESIDUALS

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ABSTRACT. *An algorithm is presented for updating finite element models based upon a minimization of dynamic residuals. The dynamic residual of interest is the force unbalance in the homogeneous form of the equations of motion arising from errors in the model's mass and stiffness when evaluated with the identified modal parameters. The present algorithm is a modification and extension of a previously-developed Sensitivity-Based Element-By-Element (SB-EBE) method for damage detection and finite element model updating. In the present algorithm, SB-EBE has been generalized to minimize a dynamic displacement residual quantity, which is shown to improve test-analysis mode correspondence. Furthermore, the algorithm has been modified to include Bayesian estimation concepts, and the underlying nonlinear optimization problem has been consistently linearized to improve the convergence properties. The resulting algorithm is demonstrated via numerical and experimental examples to be an efficient and robust method for both localizing model errors and estimating physical parameters.*

NOMENCLATURE

$[M]$, $[C]$, $[K]$	Nominal mass, damping and stiffness matrices
ω_{Ei}	Experimental frequency (rad/s) for mode i
$\{\phi_{Ei}\}$	Experimental mode shape vector for mode i
$\{R_i\}$	Dynamic residual (modal force) vector
$[Z_i]$	Undamped impedance matrix for mode i
$[\,]_m$, $[\,]_o$	Measured, unmeasured partitions of $[\,]$
$[P_i]$	Mode shape projection operator
$[B_i]$	Parameter sensitivities for mode i
J, $\{g\}$, $[G]$	Objective function, linearized gradient and Hessian
$[Q_i]$	Approximate covariance matrix of dynamic residual
$[Q_{\phi_{mi}}]$	Covariance matrix of measured mode shape
$[Q_{\omega_i^2}]$	Covariance matrix of the experimental eigenvalues
$[Q_P]$	Covariance matrix of the initial parameters
MAC_{ij}	Modal assurance criteria

1. INTRODUCTION

A significant amount of research in structural dynamics system identification has focused on methods for reconciling finite element models of structures with modal parameters identified from dynamic testing. Early approaches to this problem involved the direct updating of assembled stiffness and mass matrices to correlate to the available modes and mode shapes identified from test. In order to choose a particular solution from an infinite number of possible solutions, some quantity, such as the norm of the matrix adjustment, was minimized. Recent modifications to this general approach involve retaining the connectivity pattern of the model through constraints, or minimizing the rank of the matrix update. These methods are efficient and have been used successfully for both model adjustments and for structural damage detection. For this discussion, this class of methods can be termed optimal matrix updating.

A fundamentally different approach involves estimating or updating the "physical" parameters of the structural design, such as cross-sectional areas, elastic moduli, or added masses, used in the finite element model definition. There are a number of advantages to such an approach over optimal matrix updating methods. First, the formulation of the initial model, including its connectivity, is implicitly preserved. This is quite important since the original model, if formulated by a skilled analyst, contains a significant amount of engineering judgement about the structure of interest. Such judgement supplements the reliable, yet incomplete, knowledge gained from experimental data. Secondly, results of model updating can be understood in terms of errors in design parameters or modeling assumptions. This provides a mechanism, at least ideally, for learning and improving the future modeling of similar structures. Finally, the updated model is more generally useful for design sensitivity analysis as it retains the flexibility of the finite element method, rather than being simply a set of equations which predict the limited dynamic measurements. This approach is termed sensitivity-based model updating, as it utilizes the sensitivity of predicted and estimated quantities, such as modal parameters or response functions, to the physical parameters of the model.

The present paper addresses the problem of sensitivity-based model updating through the minimization of a dynamic residual. This residual arises due to errors in the model stiffness and mass matrices and is a reflection of the difference between the model's predicted modal parameters and the modal parameters from experimental data. It is a different approach, however, from directly comparing the predicted and measured modal parameters and does not require the computation of the model modes and determination of the correspondence between the model modes and the test modes.This is a distinct advantage, particularly when such correspondence is difficult to establish. The present algorithm is a modification and extension of a previously-developed Sensitivity-Based Element-By-Element (SB-EBE) method for finite element model updating [1].

The modifications of the basic SB-EBE algorithm are intended to address a number of practical issues encountered when applying the algorithm to complex structures. First, a consistent linearization of the governing minimization problem is derived to improve the rate of convergence of the algorithm. The new linearization couples the mode shape projection and parameter estimation stages of the algorithm at a minor computational cost, and improves the estimate of curvature in the optimization space. Secondly, the residual governing the update problem is redefined as a displacement, rather than force, quantity through a flexibility weighting. It is shown that this weighting improves the correspondence of test and analysis modal parameters typically used to assess the model's accuracy. Finally, Bayesian estimation is incorporated to condition the update problem. Bayes estimation involves the use of relative confidence measures for the parameters being updated and the observed data used to guide the estimation. This important modification leads to a more reliable algorithm, especially in the presence of small sensitivity coefficients, large model errors, and correlation between parameters.

The remainder of the paper is organized as follows. In Section 2, the basic SB-EBE theory and algorithm is reviewed. In Section 3, the new modified algorithm is developed theoretically and its implementation is detailed in Section 4. Numerical and experimental results are given in Section 5, and Section 6 offers concluding remarks.

2. REVIEW OF BASIC THEORY AND ALGORITHM

The governing equations for linear time-invariant structural dynamics are given as

$$M\ddot{q} + C\dot{q} + Kq = \hat{B}u \tag{1}$$

where K, C and M are the stiffness, damping, and mass matrices from the finite element model, q is a vector of displacements, u is a vector of applied forces, and $\hat{B}$ maps those forces to the associated degrees of freedom of the model. The homogeneous form of Eqn. 1 leads to the following undamped generalized eigenproblem:

$$K\phi = \lambda M\phi \tag{2}$$

where λ is the eigenvalue, which is equal to ω^2, the square of the undamped natural frequency, and ϕ is the associated eigenvector, which is the physically the normal (i.e. undamped) mode shape.

The basic theory, developed by Hemez and Farhat (Univ. of Colorado, 1993), determines the change Δp to a set of physical parameters of the model which minimize the norm of the dynamic force residual, viz.

$$\min_{\Delta p}\left(\sum_i \|R_i\|_2^2\right) \tag{3}$$

R_i is the dynamic force residual for mode i, defined as

$$R_i = (K - \omega_{E_i}^2 M)\phi_{Ei} \tag{4}$$

where ω_{E_i} is an experimentally-determined frequency of the structure for mode i, and ϕ_{Ei} is the associated mode shape. Unfortunately, the degrees of freedom (DOF) at which the mode shape is sampled from test is typically much smaller than the number of DOF in the finite element model which defines K and M. Therefore, to apply Eqn. 4, either the model must be reduced to the measurement DOF, or the measured portion of the mode shape must be expanded to the displacement basis of the model. Although it is more computationally intensive, the basic algorithm uses an expansion of the mode shapes to compute the dynamic residual.

The theoretical basis for correcting the model using the dynamic force residual is as follows. If the "correct" model is given as

$$\begin{Bmatrix} K_c = K + \Delta K \\ M_c = M + \Delta M \end{Bmatrix} \tag{5}$$

and from Eqn. 2

$$(K_c - \omega_{E_i}^2 M_c)\phi_{Ei} = 0 \tag{6}$$

then

$$-R_i = (\Delta K - \omega_{E_i}^2 \Delta M)\phi_{Ei} \tag{7}$$

Hence, R_i is a function of both magnitudes and locations of the model errors [2-3]. The basic Hemez algorithm consists of three key steps: mode shape projection, error localization (parameter selection), and parameter estimation.

2.1 Mode Shape Projection

To derive the proper projection operator from Eqn. 3, we must partition the mode shape ϕ_i into its measured and unmeasured components, and also partition the associated columns of the mass and stiffness matrices. Then

$$\begin{aligned} R_i &= (K - \omega_{E_i}^2 M)\phi_{Ei} \\ &= \left(\begin{bmatrix} K_m & K_o \end{bmatrix} - \omega_{E_i}^2 \begin{bmatrix} M_m & M_o \end{bmatrix}\right)\begin{Bmatrix} \phi_{Em_i} \\ \phi_{o_i} \end{Bmatrix} \end{aligned} \tag{8}$$

where ϕ_{Em_i} is the mode shape for mode i at the measurement DOF, ϕ_o is the unmeasured portion of the same mode shape, and K_m, M_m, K_o, and M_o are the measured and unmeasured column sets of the stiffness and mass matrices. The mode shape projection directly results from minimizing the dynamic residual with respect to ϕ_o, assuming no change in the model parameters, viz.

$$\begin{aligned} &\min_{\phi_o} R_i^T R_i \\ \therefore \phi_{o_i} &= -(Z_{o_i}{}^T Z_{o_i})^{-1} Z_{o_i}{}^T Z_{m_i}\phi_{Em_i} \\ &= P_{oi}\phi_{m_i}^E \end{aligned} \tag{9}$$

where

$$\begin{bmatrix} Z_{m_i} & Z_{o_i} \end{bmatrix} \begin{Bmatrix} \phi_{Em_i} \\ \phi_{o_i} \end{Bmatrix} = R_i$$
$$Z_i = K - \omega_{E_i}^2 M \tag{10}$$

Z_i can be termed the impedance or dynamic stiffness of mode i.

After the projection operator for mode i is determined, the mode shape is projected and the dynamic force residual R_i with respect to the model DOF can be computed. Recalling Eqn. 7, the DOF exhibiting the largest force residuals will be associated with the set of model elements whose parameters are significantly in error. Therefore, it is reasonable to select those parameters which cause the largest perturbations to the element matrices associated with a set of model DOF j, where $R(j)$ is above some threshold level. In the original SB-EBE method this process is termed "zooming."

2.2 Parameter Estimation

The final step, after projecting the mode shapes and choosing which model parameters to vary, is to compute the updated parameter values which minimize the sum of dynamic force residuals over a set of modes, viz.

$$\min_{\Delta p} \sum_i R_i^T R_i$$
$$\therefore \quad \sum_i B_i^T B_i \Delta p = -\sum_i B_i^T R_i \tag{11}$$

where

$$B_i = \begin{bmatrix} \frac{\partial Z_i}{\partial \Delta p_1} \phi_i & \frac{\partial Z_i}{\partial \Delta p_2} \phi_i & \dots & \frac{\partial Z_i}{\partial \Delta p_{np}} \phi_i \end{bmatrix}$$
$$R_i = (K - \omega_{E_i}^2 M) \phi_i \tag{12}$$

Here, B_i is the sensitivity of R_i to the parameters being updated.

3. NEW ALGORITHM: THEORY

The motivation for developing a new algorithm based upon the SB-EBE method came from tests of that algorithm on a moderately simple beam structure which will be reviewed in a later section. These tests revealed a number of potential problems, including unusually small magnitude parameter updates leading to slow convergence, and convergence to poor solutions as measured by relative frequency error and mode shape correlations.

Based on the above concerns, the basic theory and algorithm was reworked to incorporate:

- Consistent linearization of the optimization problem to include coupling between mode projection and parameter estimation
- Generalization of the modal error vector
- Inclusion of Bayesian estimation concepts; e.g. weighting by the force error covariance, incorporation of parameter confidence

We now proceed to detail these modifications.

3.1 Consistent linearization of the optimization problem

The solution proposed by the basic algorithm is staggered in the following sense. Although the model is being adjusted in the overall procedure, this adjustment is ignored in the determination of the mode shape projection. While this simplifies the theory somewhat, it may introduce a serious computational cost. This is because, by ignoring the coupling between the projection and the parameter estimation, the curvature of the parameter space is poorly estimated. The result is that the curvature is artificially larger, leading to smaller parameter changes and much slower convergence.

This problem can be alleviated by adding a correction to the projected partition of the mode shapes which accounts for its dependence on the parameter estimation problem. Using

$$\phi_{oi} = -(Z_{oi}^T Z_{oi})^{-1} Z_{oi}^T Z_{mi} \phi_{mi} + \delta\phi_{oi} \tag{13}$$

the linearization of the first-order conditions for Eqn. 3 leads to the following coupled system of equations:

$$G \Delta x = -g \tag{14}$$

where

$$\Delta x = \begin{Bmatrix} \Delta p \\ \delta\phi_{o1} \\ \delta\phi_{o2} \\ \vdots \\ \delta\phi_{oN} \end{Bmatrix} \qquad g = \begin{Bmatrix} \sum_i B_i^T R_i \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$
$$G = \begin{bmatrix} \sum_i B_i^T B_i & c_1 & c_2 & \dots & c_N \\ c_1^T & Z_{o1}^T Z_{o1} & 0 & \dots & 0 \\ c_2^T & 0 & Z_{o2}^T Z_{o2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_N^T & 0 & 0 & \dots & Z_{oN}^T Z_{oN} \end{bmatrix} \tag{15}$$

Reducing the system via Schur complements, we determine

$$\bar{G} \Delta p = -\bar{g} \tag{16}$$

where

$$\bar{G} = \sum_i (B_i^T B_i - c_i (Z_{oi}^T Z_{oi})^{-1} c_i^T) \qquad \bar{g} = \sum_i B_i^T R_i$$
$$c_i = B_i^T Z_{oi} + \begin{bmatrix} \frac{\partial Z_{oi}}{\partial \Delta p_1} R_i & \frac{\partial Z_{oi}}{\partial \Delta p_2} R_i & \dots & \frac{\partial Z_{oi}}{\partial \Delta p_{np}} R_i \end{bmatrix} \tag{17}$$

Comparing Eqn. 11 to Eqn. 17, it is seen that the effect of the consistent linearization is to reduce the positive curvature of the design space. This formulation introduces only a modest increase in computations as the factorization of $Z_{oi}^T Z_{oi}$ is already computed during the mode shape projection step and thus can be saved for use in Eqn. 16. The introduction of this consistent linearization, however, dramatically improves the rate of convergence of the algorithm, as will be shown in the numerical example problem.

3.2 Generalization of the modal error vector

The functional selected for the optimization is not the only clear choice for performing model update. Its advantage is that it does not require solving for the modes of the model, and tracking those analytical modes with respect to the test modes. Its disadvantage is that the updated model may not improve the errors between the analysis and test frequencies, or improve the correlation of the mode shapes. In fact, these accuracy indicators may be significantly degraded, and the resultant model cannot be judged as accurate as the initial model.

In seeking to understand the convergence of the basic algorithm to poor solutions as measured by relative frequency and mode shape errors, it is helpful to compare the technique to traditional estimation algorithms. First, we can re-write the mode i contribution to the objective function in equivalent modal parameter terms, viz.

$$R_i^T R_i \approx \alpha_i \sum_j (\omega_j^2 - \omega_{E_i}^2)^2 (MAC_{ij}) \tag{18}$$

where j ranges over all modes of the finite element model, and

$$MAC_{ij} = \frac{(\phi_j^T M \phi_{Ei})^2}{(\phi_j^T M \phi_j)((\phi_{Ei})^T M \phi_{Ei})} \qquad \alpha_i = (\phi_{Ei})^T M \phi_{Ei} \tag{19}$$

Here MAC_{ij} is the *modal assurance criterion*, which is a measure of the correlation between two mode shapes, in this case model mode j and test mode i. The parameter α_i is the modal mass of the test mode shape. From Eqn. 18, the contribution to the overall dynamic force residual from test mode i is equivalent to summing up the squares of the differences between test eigenvalue i and each of the eigenvalues of the model, which are scaled to the correlation between the test model shape and the associated model mode shape. If small correlations exist between the test mode shape and any model mode shapes with vastly different frequencies, the contribution of that term can dominate the performance index. This has the effect of biasing the algorithm away from reconciling test and model modes which correspond more closely in both mode shape and frequency.

To alleviate this problem, we can replace the modal force error by a generalized modal error $\overline{R}_i = W R_i$, where

$$\overline{R}_i^T \overline{R}_i \approx \alpha_i \sum_j \left(\frac{\omega_j^2 - \omega_{E_i}^2}{\omega_j^2} \right)^2 (MAC_{ij}) \tag{20}$$

This result can be obtained approximately by defining W as

$$W = K^{-1} \text{ or } M^{\frac{1}{2}} K^{-1} \tag{21}$$

which implies that the generalized modal error is a dynamic displacement residual quantity, rather than a dynamic force residual.

3.3 Including Bayesian estimation concepts

Although the parameters being estimated usually evolve from some nonzero initial estimate, the basic algorithm places no relative confidence on these initial values with respect to the test data used for model adjustment. The quantitative result is that there is no penalty placed on the magnitude of the parameter change. Therefore, any final parameter value, no matter what magnitude or sign, is judged as superior to the original estimate as long as the sum of the dynamic force residuals have been reduced. In actuality, there are usually both hard constraints placed on the parameter values and some degree of confidence in the initial parameter estimates. Furthermore, the test data used for model adjustment is often imperfect, and the confidence in the data varies depending on whether frequency or mode shape component estimates are being considered.

A popular approach in estimation theory to address the aforementioned concerns is the use of Bayesian estimation [4]. For linear structural dynamics applications such as the present model updating problem, Bayesian estimation reduces to a generalized least-squares problem [5]. We can modify the performance index of the basic algorithm as follows:

$$\min_{\Delta p, \phi_{oi}} J \tag{22}$$

where

$$\begin{aligned} J &= \sum_{i=1}^{N} \overline{R}_i^T Q_i^{-1} \overline{R}_i + \Delta p^T Q_p^{-1} \Delta p \\ Q_i &= \operatorname{diag}(Z_i P_i Q_{\phi_{mi}} P_i^T Z_i + Q_{\omega_i^2} M P_i (\phi_{mi} \phi_{mi}^T) P_i^T M) \end{aligned} \tag{23}$$

Q_p is the covariance matrix of the initial parameters being estimated, $Q_{\phi_{mi}}$ is the covariance matrix of the components of measured mode shape i, $Q_{\omega_i^2}$ is the variance of the square of the measured modal frequency, and P_i is the mode shape projection matrix.

The primary difficulty in introducing the Bayesian estimation concept, or equivalently a maximum likelihood estimator, is that the error quantity being minimized is not directly a measured quantity, hence the covariance being introduced is not simply the variances of the test data. Instead, the dynamic residual is a nonlinear function of the data, the model matrices and the mode shape projection, which is itself a function of the model and based on the minimization of the overall functional. Therefore, although introducing statistical measures can, in general, increase the robustness of the algorithm, the approach also leads to additional nonlinearities in which mode shape projection and modal error covariance estimates are coupled. In the present work, this nonlinearity is handled in a very cursory manner by computing an initial estimate of Q_i using only the measured component of the mode shapes. That estimate is used to compute a estimate of the mode shape projection. The mode shape projection is then used to re-compute a better estimate of Q_i, which is then used to complete the algorithm. This is basically a predictor-correction approach and seems to work adequately for the applications studied. Other possibilities might include using a completely different mode shape projection algorithm to compute Q_i.

4. NEW ALGORITHM: IMPLEMENTATION

In this section we review the step-by-step procedure for the new modified algorithm and discuss implementation issues.

4.1 Summary of Modified Algorithm

The following procedure represents one pass or iteration through the updating algorithm. Because of the inherent nonlinearity of the optimization, convergence to a solution can require many iterations.

- **Given**

$$K, M, p_o, Q_p$$

$$\left\{ \omega_{E_i}, \phi_{mi}^E, Q_{\omega_i^2}, Q_{\phi_{mi}}, i = 1, \ldots, N \right\}$$

$$\left\{ \frac{\partial K}{\partial p_j}, \frac{\partial M}{\partial p_j}, j = 1, \ldots, n_p \right\}$$

- **Initialize** $J = 0, \quad g = 0, \quad G = Q_p^{-1}$

- **For i=1 to N**

  - Form $Z_i = W(K - \omega_{E_i}^2 M)$ and partition into

$$Z_i = \left[Z_{mi} \; Z_{oi} \right]$$

  - Predict $Q_i = \mathrm{diag}(Z_{mi} Q_{\phi_{mi}} Z_{mi}^T + Q_{\omega_i^2} M_m (\phi_{mi} \phi_{mi}^T) M_m^T)$

  - Solve $(Z_{oi}^T Q_i^{-1} Z_{oi}) P_{oi} = -Z_{oi}^T Q_i^{-1} Z_{mi}$

  - Correct
$$Q_i = \mathrm{diag}(Q_i + Z_{oi} P_{oi} Q_{\phi_{mi}} P_{oi}^T Z_{oi}^T + Q_{\omega_i^2} M_o P_{oi} (\phi_{mi} \phi_{mi}^T) P_{oi}^T M_o^T)$$

  - Solve $(Z_{oi}^T Q_i^{-1} Z_{oi}) \phi_{oi} = -Z_{oi}^T Q_i^{-1} Z_{mi} \phi_{Emi}$

  - Compute $R_i = Z_i \phi_i$, $B_i = \left[b_{i1} \; b_{i2} \; \ldots \; b_{in_p} \right]$ where

$$b_{ij} = W\left(\frac{\partial K}{\partial p_j} - \omega_{E_i}^2 \frac{\partial M}{\partial p_j} \right) \phi_i$$

  - Compute $c_i = \left[c_{i1} \; c_{i2} \; \ldots \; c_{inp} \right]$, where

$$c_{ij}^T = b_{ij}^T Q_i^{-1} Z_{oi} + R_i^T Q_i^{-1} W \left(\frac{\partial K_o}{\partial p_j} - \omega_{E_i}^2 \frac{\partial M_o}{\partial p_j} \right)$$

  - Solve $(Z_{oi}^T Q_i^{-1} Z_{oi}) d_i = c_i$

  - Sum:

$$\begin{cases} J = J + R_i^T Q_p^{-1} R_i \\ g = g + B_i^T Q_i^{-1} R_i \\ G = G + B_i^T Q_i^{-1} B_i - \beta c_i^T d_i \end{cases}$$

- **Solve** $G \Delta p = -g$

- **Update**

$$K = K + \sum_j \left(\frac{\partial K}{\partial p_j} \right) \Delta p_j \qquad p = p_0 + \Delta p$$

$$M = M + \sum_j \left(\frac{\partial K}{\partial p_j} \right) \Delta p_j \qquad W = W(K, M)$$

4.2 Control of Curvature Estimate

As mentioned in the preceding sections, a consistent linearization of the optimization is employed in the modified algorithm to improve its convergence properties. Caution must be exercised, however, as this linearization does not guarantee a positive-definite hessian. This is problematic because, without active intervention by the user, the algorithm will converge to a point of maximum error rather than minimum error when in a region of the parameter space with negative curvature. The present procedure offers two mechanisms for controlling the curvature to avoid this result. The first is the use of Bayes estimation, which conditions the estimation problem by contributing a penalty term on the change in the parameter estimates. Numerically, this term provides a positive-definite contribution to the hessian which can be adjusted to reflect the analyst's relative confidence in the initial parameter estimates.

The second mechanism for controlling the curvature estimate is through the use of a constant β which parameterizes the linearization between that of the basic algorithm $(\beta = 0)$ and the modified algorithm $(\beta = 1)$. This constant controls the contribution in the linearization of the coupling between the mode shape projection and the parameter estimation. This can be seen in the detailed procedure above, where the contribution to the curvature of mode i is added to the hessian matrix G. Note that the basic algorithm is always guaranteed positive-definite, but that guarantee comes at the cost of a poorer estimate of the curvature. The use of the parameter β allows that cost to be controlled optimally by the user.

4.3 Model Reduction

Rather than projecting the mode shapes, reduction of the model to the measurement degrees of freedom can be employed. This is often avoided because the reduction of a complex model down to the small number of DOF actually measured will introduce errors in the predictive accuracy of the model which lead to nonzero dynamic residuals and hence inappropriate parameter corrections. A possible compromise is to employ a component mode synthesis type of reduction such as the Craig-Bampton technique, which augments a static condensation to the measurement DOF with a set of generalized DOF spanning the lowest eigenmodes of the dynamics omitted in the static condensation. Typically, the addition of a small number of generalized DOF is sufficient to ensure that the reduced model can predict the eigenmodes of the full-order system. The experimental modes would then be projected into this slightly larger subspace.

4.4 Statistical Significance of the Parameter Estimates

An advantage of Bayes estimation is that it allows the analyst to assess the confidence intervals for the final estimates of the parameters, as a function of their initial covariances, their sensitivity to the data used in the estimation, and the covariance of that data. A linearized estimate of the covariance of the updated parameters is given by

$$\hat{Q}_p = G^{-1} = [Q_p^{-1} + \sum \{ B_i^T Q_i^{-1} B_i - c_i (Z_{oi}^T Q_i^{-1} Z_{oi})^{-1} c_i^T \}]^{-1} \quad (24)$$

evaluated at the point of convergence. From this result, the standard deviation of the parameters can be determined by taking the square root of the diagonal elements of $\hat{Q}_p$. Of course, this statistical quantity is only as valid as the covariances of the data and the initial parameters. The updated variances relative to their initial values are useful, however, in determining whether the change in parameters is significant and based on the data.

5. APPLICATIONS

5.1 Numerical Data: Planar Truss Structure

The first example from Reference [1] was chosen to test the implementation of the modified algorithm and assess its performance relative to the basic SB-EBE procedure. This example considers a free-free planar truss with 44 translational DOF, 7 of which are measured. For this comparison, the first 5 flexible modes are used to update the model, and the only parameters being updated are the elastic modulus of the two elements modified. Furthermore, the test data is assumed to be perfect (zero variance), which implies that the Bayesian covariance weights are not used. Thus the only differences between the two algorithms are the consistent linearization of the optimization problem and the weighting of the modal force error.

The results are documented in Table 1. Note that, although the use of the flexibility weighting do help to speed the convergence, it also introduces a large computational overhead, especially when the weighting matrix is full rather than sparse. Note also that updating the weighting matrix at each iteration as the stiffness was updated did not improve the convergence of the algorithm. The need for weighting the modal error vector is dictated more by the quality of the final solution when the data is imperfect than by the convergence of the algorithm. Finally, it was found that using the full extent of the consistent linearization led to a negative definite curvature which caused the algorithm to diverge. Therefore, β was reduced to 0.95, which results in the fastest convergence.

Table 1: Comparison of Convergence Using Modified Algorithm

Method	Weighting Matrix	# update iterations
Basic SB-EBE	N/A	180
Modified $\beta = 0.95$	$W = I$ (modal force error minimization)	25
Modified $\beta = 0.95$	$W = K_o^+$ (held constant)	8
Modified $\beta = 0.95$	$W = K_{up}^+$ (updated each iteration)	9

The cases documented above were based upon the same convergence criterion. The parameter results for the basic SB-EBE algorithm and the modified algorithm with $W = I$ are shown in Figure 1. Note here that, even at 180 update iterations, the basic algorithm has still not reached the correct updated parameter values, while the modified algorithm with its consistent linearization has converged to within 1% of the correct values in less than 30 iterations

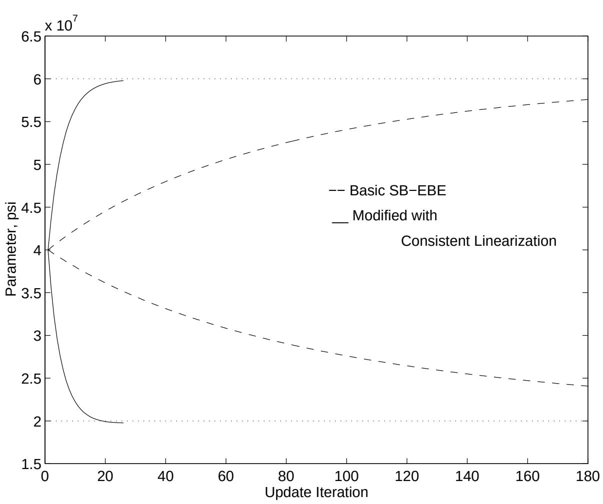

Figure 1: Convergence of Parameters for Numerical Example

5.2 Experimental Data: LADDER Structure

The experimental example problem is a tubular welded structure representative of an automotive engine support. The goal of the model updating was to determine unknown joint compliance parameters, and to adjust the basic properties, in order to correlate the first 14 modes identified from test. The test setup is shown in Figure 2.

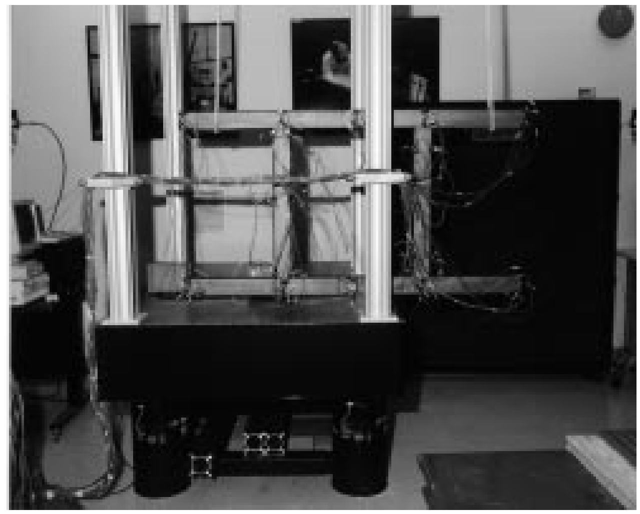

Figure 2: Modal Testing Setup for LADDER Structure

The structure was instrumented with 96 accelerometers grouped in 16 locations in order to extract both translational and rotational response at beam cross-sections throughout the structure. The finite element model of the structure is shown in Figure 3; it is a NASTRAN model consisting of CBEAM elements, with spring elements introduced to model the joint compliances. Rigid offsets were used to determine the responses at each accelerometer location, and instrumentation mass was included. The correlation of the modal parameters between the test-identified modes and the initial (pre-test) analysis model is documented in Table 2.

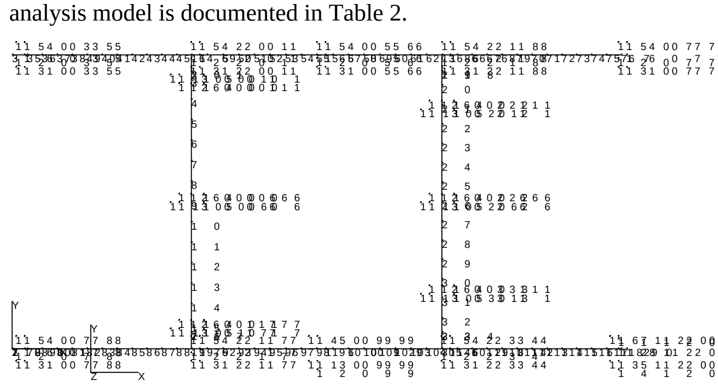

Figure 3: Finite Element Model of LADDER Structure

After attempts to reconcile the model using the basic SB-EBE procedure failed, the modified algorithm was developed and applied to this problem. The update evolved as follows: the joint spring parameters were estimated based on the first 8 flexible modes with the model statically reduced to the 96 sensor DOF. This implied that no mode shape projection was performed. Then, cross-sectional parameters I_1, I_2, J were added and estimated along with joint rotational

Table 2: Initial LADDER Model/Test Comparison

Test Mode	Test Frequency (Hz)	Model Mode	Model Frequency (Hz)	%difference Frequency	Modal Assurance Criteria
1	78.9674	1	72.3633	-8.36	0.9973
2	170.6259	3	174.9456	2.53	0.9963
3	174.4670	2	161.5404	-7.41	0.9934
4	214.7231	4	206.3898	-3.88	0.9981
5	250.9062	5	255.1062	1.67	0.9951
6	312.1717	7	318.6140	2.06	0.9580
7	315.7890	6	312.8396	-0.93	0.9516
8	317.7661	9	368.6281	16.01	0.9486
9	330.2652	8	333.6956	1.04	0.9968
10	432.5194	10	451.6765	4.43	0.9937
11	518.5953	11	534.4661	3.06	0.9890
12	563.6540	14	806.4039	43.07	0.8115
13	612.8141	12	631.6433	3.07	0.9816
14	674.3648	13	678.9766	0.68	0.7993

springs using test modes 1-9 and model statically reduced to the sensor DOF. The final values were estimated based on modes 1-12 with same parameters plus K_{uy} and using the model reduced to measured DOF + torsional DOF for model grids. This final estimation thus required that the mode shapes be projected.

The resultant parameter values are given in Table 3. The correlation of the updated model to test for the first 14 modes are documented in Table 4. Observe that the frequency errors have been reduced from a maximum of 43% to below 4%, while the mode shape correlations have been maintained or slightly improved. Note also from the parameter update results that the updated coefficients of variation (COV), which is the standard deviation of the parameter expressed as a percentage of the parameter value, is significantly smaller than the assumed initial COV. This implies that the parameters were highly sensitive to the modal data used in the estimation. In conclusion, the present modified algorithm performed very well using the experimental data, resulting in a highly accurate updated model.

Table 3: Parameter Update Results for LADDER Structure

Parameter	Final Value (relative to initial)	Initial COV	Updated COV
K_{uy}	0.4250	100%	0.49%
$K_{\theta x}$	0.2580	100%	0.00153%
$K_{\theta y}$	104.0	100%	3.58%
$K_{\theta z}$	1.4621	100%	1.39%
I_1	0.9415	3%	0.00663%
I_2	0.9178	3%	0.0191%
J	1.0091	3%	0.00661%

6. CONCLUDING REMARKS

An algorithm for updating finite element models using modal data has been presented. The algorithm minimizes a generalized dynamic residual which is a function of the experimental modal parameters and the model mass and stiffness matrices. The present algorithm is a modification of a previous method for sensitivity-based element-by-element model updating and incorporates a generalized error weighting, consistent linearization and Bayesian estimation. The algorithm has been demonstrated on numerical and experimental data and has been shown to be an efficient and effective approach for estimating parameters to reconcile test and analysis models.

Table 4: Final LADDER Model/Test Comparison

Test Mode	Test Frequency (Hz)	Model Mode	Model Frequency (Hz)	%difference Frequency	Modal Assurance Criteria
1	78.9674	1	78.8034	-0.21	0.9978
2	170.6259	2	169.6736	-0.56	0.9963
3	174.4670	3	174.6665	0.11	0.9931
4	214.7231	4	218.2671	1.65	0.9984
5	250.9062	5	249.0289	-0.75	0.9957
6	312.1717	6	307.9859	-1.34	0.9894
7	315.7890	7	315.5987	-0.06	0.9789
8	317.7661	8	323.0028	1.65	0.8792
9	330.2652	9	324.1070	-1.86	0.9521
10	432.5194	10	435.3196	0.65	0.9955
11	518.5953	11	514.9591	-0.70	0.9894
12	563.6540	12	542.8199	-3.67	0.8724
13	612.8141	13	615.0687	0.37	0.9732
14	674.3648	14	673.2796	-0.16	0.8250

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